AD-A071 406 GEORGE WASHINGTON UNIV WASHINGTON D C PROGRAM IN LOG--ETC F/G 15/5
TRANSIENT SOLUTIONS FOR REPAIRABLE ITEM PROVISIONING.(U)
APR 79 Z BARZILY, D GROSS N00014-75-C-0729

UNCLASSIFIED SERIAL-T-390 NL

1 OF 1

AD
A071406

END
DATE
FILMED
8-79
DDC

THE
GEORGE
WASHINGTON
UNIVERSITY

LEVEL II

STUDENTS FACULTY STUDY R
ESEARCH DEVELOPMENT FUT
URE CAREER CREATIVITY CO
MMUNITY LEADERSHIP TECH
NOLOGY FRONTI ESIGN
ENGINEERING APP ENC
GEORGE WASHIN NIV

INSTITUTE FOR MANAGEMENT
SCIENCE AND ENGINEERING
SCHOOL OF ENGINEERING
AND APPLIED SCIENCE

6

TRANSIENT SOLUTIONS FOR REPAIRABLE ITEM PROVISIONING

by

10

Zeev Barzily
Donald Gross

9 Scientific rept.

14

Serial-T-390
9 April 1979

11

12 36p.

The George Washington University
School of Engineering and Applied Science
Institute for Management Science and Engineering

Program in Logistics
15 Contract N00014-75-C-0729
Project NR 347 020
Office of Naval Research

405 337

JB

NONE

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER T-390	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) TRANSIENT SOLUTIONS FOR REPAIRABLE ITEM PROVISIONING		5. TYPE OF REPORT & PERIOD COVERED SCIENTIFIC 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) ZEEV BARZILY DONALD GROSS		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0729
9. PERFORMING ORGANIZATION NAME AND ADDRESS THE GEORGE WASHINGTON UNIVERSITY PROGRAM IN LOGISTICS WASHINGTON, D.C. 20037		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS OFFICE OF NAVAL RESEARCH CODE 434 ARLINGTON, VA 22217		12. REPORT DATE 9 APRIL 1979 13. NUMBER OF PAGES 32
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) NONE 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

DISTRIBUTION IS UNLIMITED; APPROVED FOR PUBLIC SALE AND RELEASE.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

FINITE SOURCE QUEUES
TRANSIENT SOLUTIONS
MACHINE REPAIR

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

It is desired to find the spares inventory level and number of repair channels necessary to guarantee a prespecified service level for a population of items that randomly fail and have exponential repair times. Steady state solutions are attainable from finite source queueing theory. This paper looks at transient effects and the speed of convergence to steady state.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

NONE

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
Institute for Management Science and Engineering
Program in Logistics

Abstract
of
Serial T-390
9 April 1979

TRANSIENT SOLUTIONS FOR REPAIRABLE ITEM PROVISIONING

by

Zeev Barzily
Donald Gross

It is desired to find the spares inventory level and number of repair channels necessary to guarantee a prespecified service level for a population of items that randomly fail and have exponential repair times. Steady state solutions are attainable from finite source queueing theory. This paper looks at transient effects and the speed of convergence to steady state.

Research Supported by
Contract N00014-75-C-0729
Project NR 347 020
Office of Naval Research

THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
Institute for Management Science and Engineering
Program in Logistics

TRANSIENT SOLUTIONS FOR REPAIRABLE ITEM PROVISIONING

by

Zeev Barzily
Donald Gross

1. Introduction

It is desired to find the level of spares inventory (y) and number of repair channels (c) to support an operating population of M items. An operating item may randomly fail and require repair. Both the time to failure and the repair time of any item are independently distributed exponential random variables with parameters λ and μ, respectively. All items are identical and independent.

Generally one is interested in knowing how many spares and repair channels (or repairmen) are required in order to support the system at a certain level measured in terms such as the probability that the population is operating at full strength or the probability that a certain percentage of the population is operational or the probability that the spares pool is not empty upon a request for a spare.

The classical finite source queueing theory (also known as the *machine repair problems*) can be used to provide the steady state distribution of the number of units in and awaiting repair which is required to calculate the service level measures. Further, using the costs of

spares and repair channels, a procedure can be developed to find the optimal combination of spares and repair channels to meet a specified level of service (see Reference [6]).

In many cases, the provisioning for spares and repair capacities must be planned in advance. Also, in many new systems, operating units are added over an initial period until the population reaches its full strength. An example of this is a gas turbine engine fleet which is to build to a full strength of 256 ships, starting with an initial fleet size of 5 the first year and adding ships over an 11-year period. When the population size changes, especially if new units have different mean times to failure or mean repair times (which is likely due to technological growth and learning), transient effects might be significant. In Reference [6], it was assumed that the population reached instantaneous steady state every year at new population values. We concentrate here on methods of analyzing transient effects so that situations can be noted for which steady state assumptions may be in doubt as well as those for which steady state assumptions may be adequate.

2. Calculation of Transient Probabilities

The state of the system is defined by the number of items in and awaiting repair, which can vary between zero and $M+y$. Letting $p_{ij}(t)$ represent the probability of the system being in state j at time t given that it started in state i ($0 \leq i,j, \leq M+y$) , $\pi_i(t)$ be the unconditional probability of the system being in state i at time t, $P(t) = \{p_{ij}(t)\}$ be the $(M+y+1) \times (M+y+1)$ matrix of transition probabilities $p_{ij}(t)$, $\pi(t)$ be the $M+y+1$ component vector whose elements are the $\pi_i(t)$, and $\pi(0)$ be the initial probability vector whose components $\pi_i(0)$ are the probabilities that the system starts in state i, we have

$$\pi(t) = \pi(0)P(t) . \tag{1}$$

Denoting the steady state probability vector by π ,

$$\pi = \lim_{t\to\infty} \pi(t) . \tag{2}$$

It is not necessary to obtain $\pi(t)$ if only π is of interest since this can be determined directly by steady state finite source queueing results (for example, see [5], pp. 118-125). However, to assess transient effects, it *is* necessary to calculate $\pi(t)$; in fact, it is of interest to compare $\pi(t)$ to π to observe the speed of convergence to steady state.

Solving for $\pi(t)$ requires solving a set of $M+y+1$ differential equations in $M+y$ unknowns. Obtaining analytical solutions is extremely difficult and in most cases impossible unless $M+y$ is very small. However, a variety of procedures are available which can yield numerical solutions to the differential equations. Reference [4] compares several of these numerical procedures and concludes that the so-called *randomization procedure* has some advantages. This procedure is presented (although not under that name) in Reference [3] on page 46.

For the purposes of study in this paper, the randomization procedure appears useful, although if solutions for $\pi(t)$ are desired for a large number of t values, the *Runge-Kutta* method may be more efficient. The randomization procedure will yield $P(t)$, from which $\pi(t)$ can be calculated using Equation (1). The calculation procedure is as follows. Let Q dencte the intensity matrix (infinitesimal generator) of the Markov chain of this birth-death system, that is

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 \ldots & & & 0 \\ \mu_1 & -\lambda_1-\mu_1 & \lambda_1 & 0 & \ldots & 0 \\ 0 & \mu_2 & -\lambda_2-\mu_2 & \lambda_2 \ldots & 0 & 0 \\ \vdots & & & \vdots & \vdots & \vdots \\ 0 \ldots & & & 0 & \ldots \mu_{M+y} & \mu_{M+y} \end{pmatrix},$$

where

$$\lambda_i = \begin{cases} M\lambda & , \quad (0 \leqq i < y) \\ (M-i+y)\lambda & , \quad (y \leqq i \leqq y+M) \\ 0 & , \quad (i \geqq y+M) \end{cases}$$

$$\mu_i = \begin{cases} i\mu & , \quad (0 \leqq i < c) \\ c\mu & , \quad (i \geqq c) \end{cases} .$$

The birth-death differential equations then can be written

$$\pi'(t) = \pi(t)Q , \tag{3}$$

where $\pi'(t)$ is the vector whose components are $\pi_i'(t) = d\pi_i(t)/dt$. The randomization procedure gives a method of calculating $P(t)$ using Q . Denoting the i,jth element of Q by q_{ij} , define a scalar α as

$$\alpha \equiv \max q_{ii} .$$

Let β be another scalar such that

$$\beta \geqq \alpha .$$

Then, form a matrix P with element p_{ij} as

$$P = \{p_{ij}\} = I + \frac{1}{\beta} Q ,$$

where I is an $(M+y+1) \times (M+y+1)$ identity matrix. It can be shown that the elements of $P(t)$ can be obtained by

$$p_{ij}(t) = e^{-\beta t} \sum_{n=0}^{\infty} \frac{\beta^n t^n}{n!} p_{ij}^{(n)} , \tag{4}$$

where $p_{ij}^{(n)}$ is the (i,j)th element of the matrix $P^{(n)}$ and $P^{(n)}$ is the matrix P raised to the power n . Thus the computations are not simple since they involve raising the matrix P to the nth power for $n=0,1,2,\ldots$. Obviously, the numerical procedure must truncate n at some appropriate value and our criterion was to truncate n at a number N , such that

$$\frac{p_{ij}(t) - p_{ij}^N(t)}{p_{ij}^N(t)} \leq \varepsilon ,$$

where

$$p_{ij}^N(t) = e^{-\beta t} \sum_{n=0}^{N} \frac{\beta^n t^n}{n!} p_{ij}^{(n)} .$$

We are fortunate here in being able to guarantee this since

$$p_{ij}(t) = e^{-\beta t} \sum_{n=0}^{\infty} \frac{(\beta t)^n}{n!} p_{ij}^{(n)} = p_{ij}^N(t) + e^{-\beta t} \sum_{n=N+1}^{\infty} \frac{(\beta t)^n}{n!} p_{ij}^{(n)} .$$

Now

$$e^{-\beta t} \sum_{n=N+1}^{\infty} \frac{(\beta t)^n}{n!} p_{ij}^{(n)} < e^{-\beta t} \sum_{n=N+1}^{\infty} \frac{(\beta t)^n}{n!} = e^{-\beta t} \left[e^{\beta t} - \sum_{n=0}^{N} \frac{(\beta t)^n}{n!} \right]$$

$$= 1 - e^{-\beta t} \sum_{n=0}^{N} \frac{(\beta t)^n}{n!} .$$

Thus

$$p_{ij}(t) - p_{ij}^N(t) < 1 - e^{-\beta t} \sum_{n=0}^{N} \frac{(\beta t)^n}{n!} .$$

Hence we can choose the smallest N such that the absolute error

$$1 - e^{-\beta t} \sum_{n=0}^{N} \frac{(\beta t)^n}{n!} \leq \varepsilon . \tag{5}$$

Because some of the $p_{ij}(t)$ may be very small, a somewhat better criterion might be to choose N such that

$$\frac{1 - e^{-\beta t} \sum_{n=0}^{N} \frac{(\beta t)^n}{n!}}{e^{-\beta t} \sum_{n=0}^{N} \frac{(\beta t)^n}{n!} p_{ij}^{(n)}} \leq \varepsilon ; \tag{6}$$

that is, the percentage error is less than or equal to ε . This is somewhat more complex to calculate however.

This procedure was programmed on an IBM 370/148. Analytical solutions are obtainable for the special case $M = y = c = 1$ for which there are only three states ($i=0,1,2$). The three differential equations in three unknowns given by Equation (3) can be solved using Laplace transforms (see Reference [1]). The numerical procedure was checked against the analytical solution for the special case and found to be quite accurate. Table I shows the comparison using criteria from Inequality (5); that is, the absolute error criterion with $\varepsilon = 10^{-3}$, and 10^{-6}. We see

TABLE I

COMPARISON OF NUMERICAL AND ANALYTICAL SOLUTIONS

$M = y = c = 1; \quad \lambda = 0.5, \quad \mu = 1.0$

Anal. (Analytical); Num. (Numerical); Num[a]: $\varepsilon = 10^{-3}$, Num[b]: $\varepsilon = 10^{-6}$

	t	$p_{00}(t)$	$p_{01}(t)$	$p_{02}(t)$
Anal.	0	1	0	0
Num[a]		1	0	0
Num[b]		1	0	0
Anal.	1	.7265	.2227	.0508
Num[a]		.7260	.2224	.0506
Num[b]		.7266	.2226	.0508
Anal.	3	.6010	.2768	.1222
Num[a]		.6003	.2767	.1222
Anal.	5	.5775	.2839	.1386
Num[a]		.5770	.2837	.1385
Anal.	10	.5715	.2857	.1428
Num[a]		.5710	.2854	.1427

[a]Running time to generate four 3×3 $P(t)$ matrices ($t=1,3,5,10$): 15.78 sec.

[b]Running time to generate one 3×3 $P(t)$ matrix ($t=1$): 3 min.

that the numerical procedure, with $\varepsilon = 10^{-3}$, is always exact to the second decimal place and in most cases to the third. While lowering ε to 10^{-6} helps somewhat on the accuracy (see the $t=1$ case in Table I), the added expense in running time becomes enormous and hence it was felt that 10^{-3} was satisfactory.

3. Results

Figure 1 shows the results of a case where $M = 10$, $y = 4$, $c = 2$, $\lambda = .15$, $\mu = 1.0$. The measure of system service used is the probability that the entire population is operational. Denoting this by $A(t)$, we have

$$A(t) = \sum_{i=0}^{y} \pi_i(t) .$$

Two cases are shown, one in which the initial conditions had all machines "up," that is, no machines in or awaiting repair, one machine operating, and one in the spares pool [$\pi(0)=(1,0,0)$], and the other where both machines were "down," that is, one in repair and the other in the repair queue [$\pi(0)=(0,0,1)$]. We can see that in the first case steady state is approached rather quickly while in the second the approach is considerably slower.

There are two factors influencing the speed to steady state. One is the initial conditions, that is, how far the system is from steady state initially, and the other is the speed at which the transient terms (terms depending on t) of the full solution to the differential equations damp out. In some cases it is known what the initial conditions are or are likely to be, but in many cases information about the initial state of the system may not be available or reliable. In order to divorce the effect of the initial conditions from how quickly the transient terms damp out, we decided to look at the $P(t)$ matrix itself. We became interested in two facets of the $P(t)$ matrix. First, we wished to obtain information about the behavior of the $p_{ij}(t)$ as t increases and second, we wished to determine for a given t how "far" $P(t)$ was from its steady state value $P(\infty)$, where

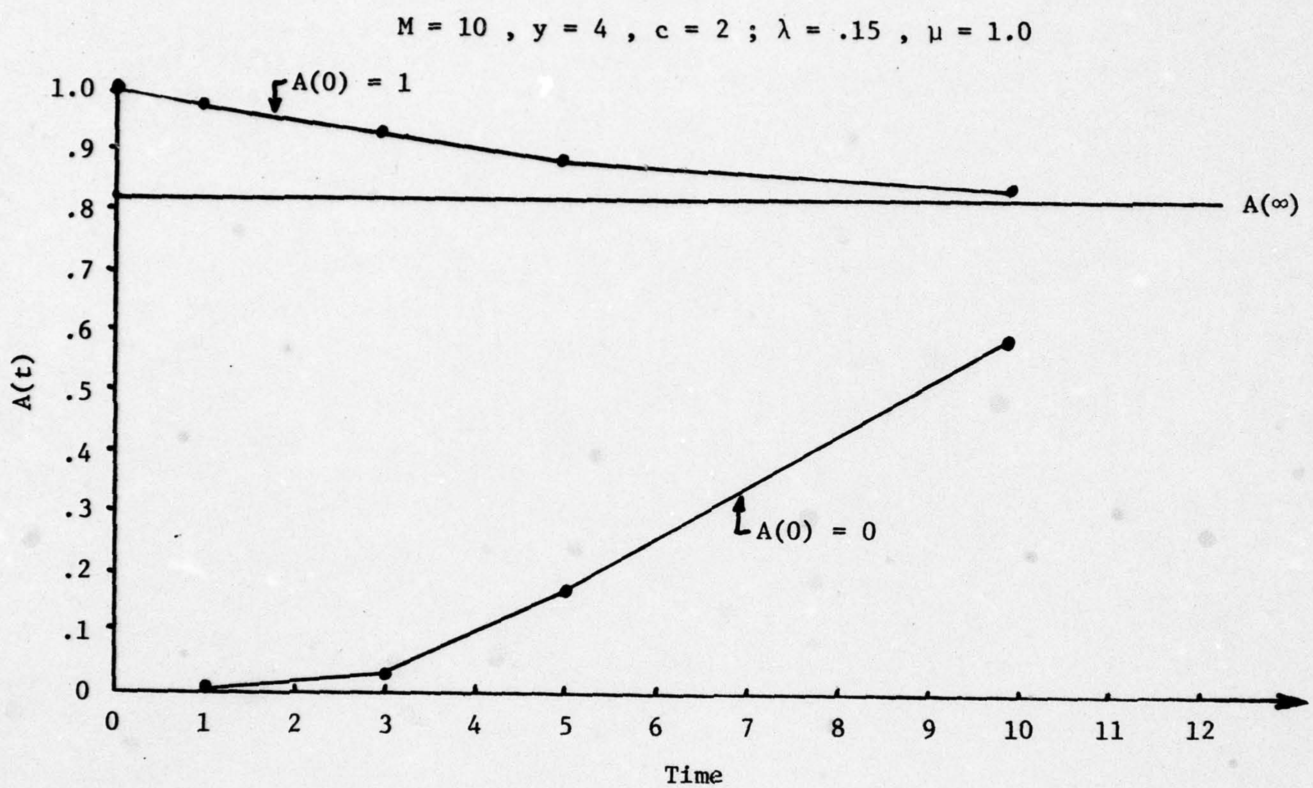


Figure 1.--Example of $A(t)$ versus t .

$$P(\infty) = \begin{pmatrix} \pi_0 & \pi_1 & \cdots & \pi_{M+y} \\ \pi_0 & \pi_1 & \cdots & \pi_{M+y} \\ \vdots & & & \\ \pi_0 & \pi_1 & \cdots & \pi_{M+y} \end{pmatrix},$$

that is, a matrix in which all rows are the steady-state probability vector π . This necessitated the development and study of single value measures for the "distance" between $P(t)$ and $P(\infty)$.

3.1 Behavior of $p_{ij}(t)$

We know $P(0)$ is an $(M+y+1) \times (M+y+1)$ identity matrix and $P(\infty)$ is the $(M+y+1) \times (M+y+1)$ matrix with all rows equal to the vector π . We are interested in what happens in between. For the case $M = 10$, $y = 4$, $c = 2$, $\lambda = .05$, $\mu = 1.0$, we plotted the first four probabilities of row three of the matrices $P(t)$, $t=0,1,3,5,10$ and ∞ ; that is, we plotted $p_{20}(t)$, $p_{21}(t)$, $p_{22}(t)$, and $p_{23}(t)$ for the above values of t (see Figure 2).

It can be seen that not all of the $p_{ij}(t)$'s approach their steady state value monotonically. The diagonal element $p_{22}(t)$ and the first element p_{20} do, $p_{22}(t)$ from above and $p_{20}(t)$ from below. But the other elements, $p_{21}(t)$ and $p_{23}(t)$, start at zero, overshoot their steady state values, and then appear to monotonically decrease toward them. We will show that $p_{ii}(t)$ will always approach π_i from above and, although it has not been proven, we conjecture that if $p_{ij}(t_1) \geqq \pi_j$, then $p_{ij}(t) \geqq \pi_j$ for all $t \geqq t_1$; that is, $p_{ij}(t)$, $i \neq j$, starts at zero, overshoots π_i *at most once*, and then approaches π_j from above.

The above conjecture can be argued by the following physical analogy. Consider a row of the $P(t)$ matrix as a bar of water. At $t=0$, a sugar cube is dropped in at the diagonal (i,i) position. As time passes,

$M = 10$, $y = 4$, $c = 2$; $\lambda = .05$, $\mu = 1.0$

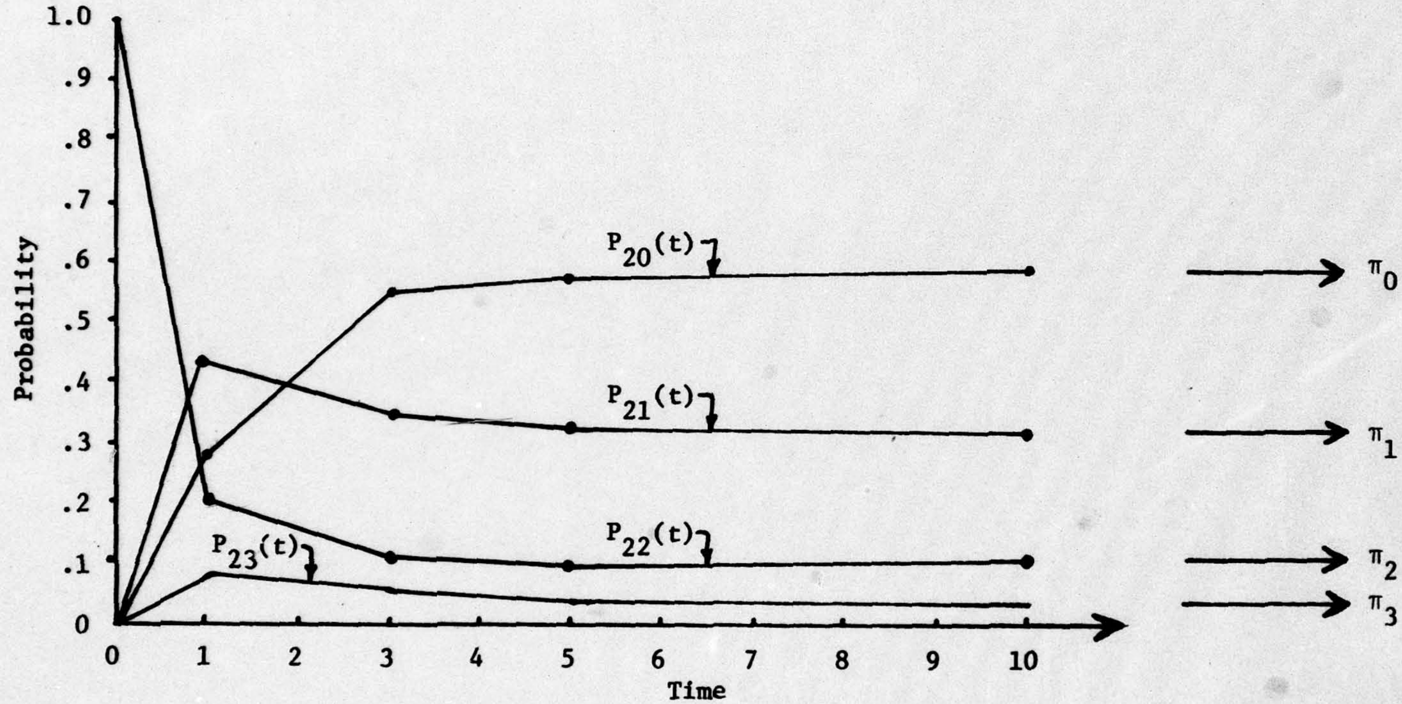


Figure 2.--Behavior of $p_{ij}(t)$ versus t .

the sugar is dispersed along the row until it reaches its steady-state concentration π_i $(i=0,1,2,\ldots,M+y)$. Note that at i,i the sugar gradually flows away from its initial concentration (of 1) to its final concentration π_i . At either end, sugar gradually flows in until it builds up to the steady state concentrations π_0 and π_{M+y} . All other states start at zero, receive sugar from the diagonal element, some of which will eventually flow past to other states and some of which will remain, causing the overshoot prior to settling down to its steady-state concentration. It is possible that other states near to the "end" of the row may not overshoot depending on the rates of sugar flow in versus sugar flow out. We expect the greatest overshoots to be in the states near the diagonal element.

We next looked at the behavior of the $p_{ij}(t)$ as a function of j for fixed t . That is, it was of interest to observe which $p_{ij}(t)$ were larger than their steady state values, π_i , and which were smaller, for a given value of t . In all cases run, we observed one of three situations as shown in Figure 3. It can be seen that the $p_{ij}(t)$'s which overestimate their π_j's appear together in a "clump." If this were always true, there could be significant savings in the computational effort required for obtaining the distance measures $(|P(\infty)-P(t)|)$ discussed in the next section. We have proved that this clumping will always be the case as well as its relative position in the row in the lemma and theorems presented below.

Lemma: Let $N_i(t)$ = the number of customers in and awaiting repair at time t given i in and awaiting repair at time 0. Then

$$\Pr\left(N_{i+1}(t) \leqq k\right) \leqq \Pr\left(N_i(t) \leqq k\right) \leqq \Pr\left(N_{i+1}(t) \leqq k+1\right) .$$

Proof: Let

$$\tau_i = \min\{u: N_i(u) = N_{i+1}(u) , 0 \leqq u \leqq \infty\} .$$

It is obvious that

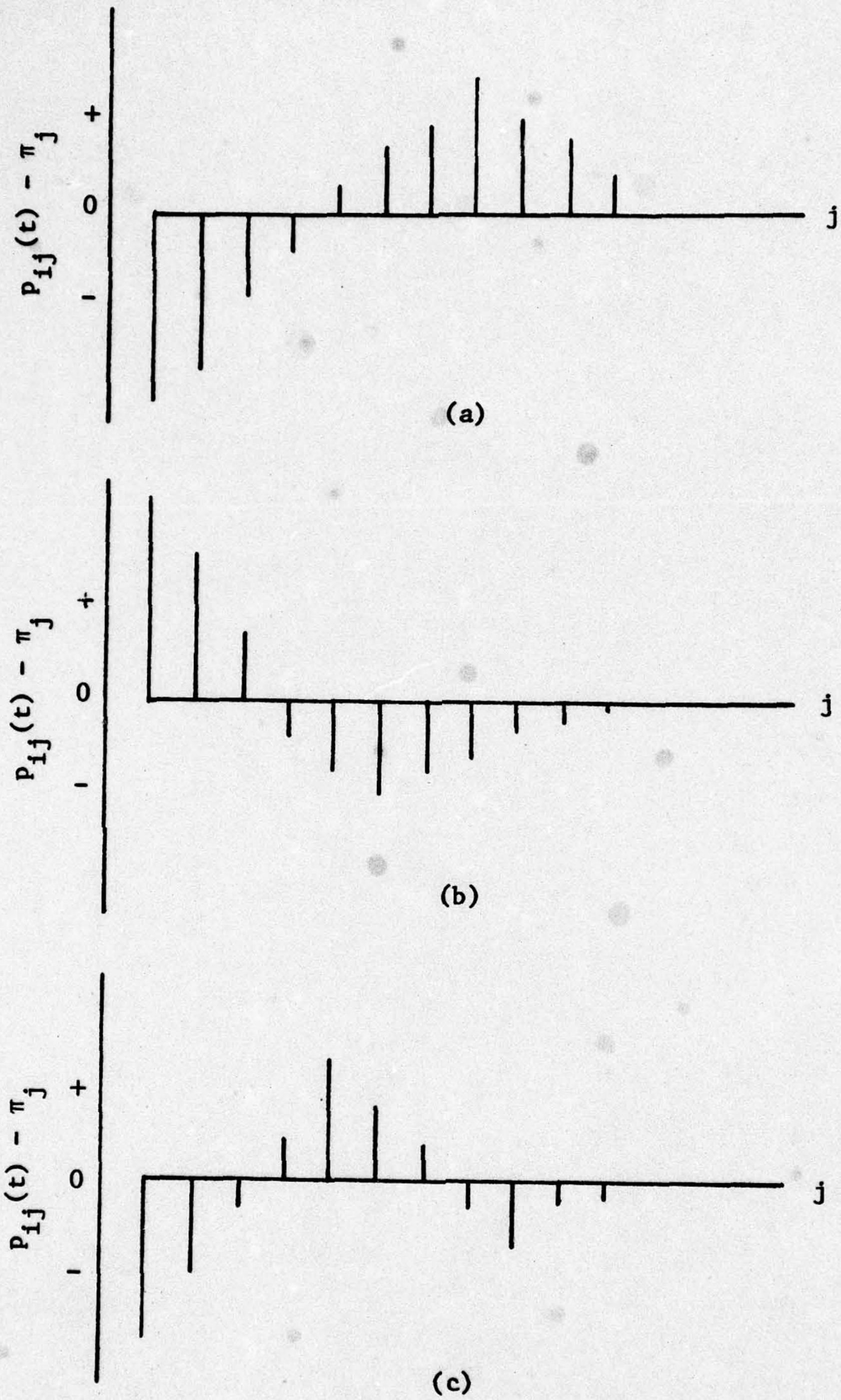


Figure 3.--Behavior of $p_{ij}(t)$ versus j , t fixed.

$$\Pr[N_i(t)=k \mid \tau_i \leq t] = \Pr[N_{i+1}(t)=k \mid \tau_i \leq t], \quad k=0,1,2,\ldots, \tag{7}$$

$$\Pr[N_i(t) \leq k \mid \tau_i > t] \geq \Pr[N_{i+1}(t) \leq k \mid \tau_i > t]. \tag{8}$$

Combining (7), (8), and the fact that $\Pr(\tau_i > t) > 0$ yields

$$\Pr\{N_i(t) \leq k\} \geq \Pr\{N_{i+1}(t) \leq k\}. \tag{9}$$

Now we denote by $A_i(t)$ the number of arrivals in $[0,t]$ and by $S_i(t)$ the number of service completions in the same time interval, given i in system at $t=0$. Since $\lambda_j \geq \lambda_{j+1}$ and $\mu_j \leq \mu_{j+1}$ for all $j \geq 0$, then

$$\Pr\{A_i(t) - S_i(t) \leq k\} \leq \Pr\{A_{i+1}(t) - S_{i+1}(t) \leq k\}, \quad \text{for any } k.$$

The proof is now complete since

$$N_j(t) = j + A_j(t) - S_j(t).$$

From (9) we obtain that

$$p_{00}(t) \geq p_{10}(t) \geq p_{20}(t) \geq \cdots,$$

and as

$$\pi_0 = \sum_{i=0}^{M+Y} \pi_i p_{i0}(t),$$

then

$$p_{00}(t) \geq \pi_0.$$

The next theorem extends this result for all $p_{ii}(t)$.

Theorem 1: $p_{ii}(t) > \pi_i \ \forall$ i and t.

Proof: We compare our system to a Markovian queueing system with only two states, 0 and 1. Denote the rate out of state 0 by ν and the rate out of state 1 by η. Let $r_{0,i}(t)$ be the transient probability of being in state i (i=0,1) at t given that the system started in state 0. The steady state probabilities of the two-state system depend on the ratio ν/η. We choose ν/η such that

$$\lim_{t\to\infty} r_{00}(t) = \pi_i \ .$$

It is well known that $r_{00}(t) \geqq \pi_i$ for any ν , η , and t ; hence the required result is obtained if η and ν can be chosen to yield

$$r_{00}(t) \leqq p_{ii}(t) \ . \tag{10}$$

Since $r_{00}(0) = p_{ii}(0) = 1$, then Equation (10) is satisfied if

$$r'_{00}(t) \leqq p'_{ii}(t) \tag{11}$$

at any point t where

$$r_{00}(t) = p_{ii}(t) \ . \tag{12}$$

To determine the conditions under which Inequality (11) holds, we write

$$\begin{aligned} p'_{ii}(t) \;=\; p'_{ii}(t) - p'_{ii}(\infty) \;=\; & -(\lambda_i+\mu_i)\left(p_{ii}(t)-\pi_i\right) \\ & + \lambda_{i-1}\left(p_{i,i-1}(t)-\pi_{i-1}\right) + \mu_{i+1}\left(p_{i,i+1}(t)-\pi_{i+1}\right) \end{aligned} \tag{13}$$

and

$$r'_{00}(t) \;=\; r'_{00}(t) - r'_{00}(\infty) \;=\; -(\nu+\eta)\left(r_{00}(t)-\pi_i\right) \ . \tag{14}$$

Using Equation (12) in (13) and then comparing (13) to (14) yields that Inequality (11) holds iff

$$-(\nu+\eta) \leqq \frac{-(\lambda_i+\mu_i)\left(p_{ii}(t)-\pi_i\right) + \lambda_{i-1}\left(p_{i,i-1}(t)-\pi_{i-1}\right) + \mu_{i+1}\left(p_{i,i+1}(t)-\pi_{i+1}\right)}{p_{ii}(t) - \pi_i} \tag{15}$$

The RHS of (15) is clearly bounded since the matrix $P(t)$ can be expressed as

$$P(t) \;=\; \lim_{t\to\infty} P(t) + B_2 e^{C_2 t} + B_3 e^{C_3 t} + \dots + B_{M+Y} e^{C_{M+Y} t} \ ,$$

where $C_i < 0$ is the ith eigenvalue of the infinitesimal generator matrix Q and B_i is a $(M+Y+1) \times (M+Y+1)$ matrix that has finite elements (see, for example, [2], pp. 364-368). Thus we conclude that ν

and η can be chosen big enough so that (11) is satisfied at any point at which (12) is satisfied, and the proof is now complete.

Theorem 2: For any i and t there exists a $j_\ell(i,t)$ and a $j_r(i,t)$, $j_\ell(i,t) \leqq i \leqq j_r(i,t)$, such that

$$p_{ij}(t) - \pi_j > 0\ , \quad \text{if } j = j_\ell(i,t) \text{ or } j = j_r(i,t)$$

$$p_{ij}(t) - \pi_j \geqq 0\ , \quad \text{if } j_\ell(i,t) < j < j_r(i,t)$$

and

$$p_{ij}(t) - \pi_j \leqq 0\ , \quad \text{otherwise.}$$

Proof: Denote

$$K(i,t) = \{k\colon p_{ik}(t) = \pi_k\}\ .$$

At $t=0$ we have $p_{ii}(0) = 1$ and $p_{ij}(0) = 0$, $i \neq j$; hence the required conditions are satisfied. To complete the proof we have to show that for any $k \in K(i,t)$ we have

$$p'_{ik}(t) \leqq 0\ , \quad \text{if } k < j_\ell(i,t)-1 \text{ or } k > j_r(i,t)+1$$

$$p'_{ik}(t) \geqq 0\ , \quad \text{if } j_\ell(i,t) < k < j_r(i,t)\ .$$

To show that the above conditions hold, we write

$$p'_{ik}(t) = p'_{ik}(t) - p'_{ik}(\infty) =$$

$$\begin{cases} -\lambda_k\big(p_{ik}(t)-\pi_k\big) + \mu_{k+1}\big(p_{i,k+1}(t)-\pi_{k+1}\big) & , k=0 \\ -(\lambda_k+\mu_k)\big(p_{ik}(t)-\pi_k\big) + \lambda_{k-1}\big(p_{i,k-1}(t)-\pi_{k-1}\big) + \mu_{k+1}\big(p_{i,k+1}(t)-\pi_{k+1}\big) & , M+Y-1 > k > 0 \\ -\mu_k\big(p_{ik}(t)-\pi_k\big) + \lambda_{k-1}\big(p_{i,k-1}(t)-\pi_{k-1}\big) & , k=M+Y \end{cases} \tag{16}$$

If $k < j_\ell(i,t)-1$ or $k > j_r(i,t)+1$, then $p_{i,k-1}(t) - \pi_{k-1} \leqq 0$ and $p_{i,k+1}(t) - \pi_{k+1} \leqq 0$; thus $p'_{ik}(t) \leqq 0$. If $j_\ell(i,t) < k < j_r(i,t)$, then $p_{i,k-1}(t) - \pi_{k-1} \geqq 0$ and $p_{i,k+1}(t) - \pi_{k+1} \geqq 0$; thus $p'_{ik}(t) \geqq 0$.

3.2 Measures of the distance between $P(t)$ and $P(\infty)$

We now turn our interest to gaining information concerning how far from steady state the system is at a given time. We desire some measure of the distance between two matrices $P(t)$ and $P(\infty)$; that is, a measure of $|P(t) - P(\infty)|$.

Consider the following example for $P(t_1)$, $P(t_2)$, and $P(\infty)$:

$$P(t_1) = \begin{pmatrix} .8 & .2 & 0 \\ .6 & .3 & .1 \\ 0 & .6 & .4 \end{pmatrix}, \quad P(t_2) = \begin{pmatrix} .3 & .5 & .2 \\ .2 & .4 & .4 \\ .1 & .5 & .4 \end{pmatrix},$$

$$P(\infty) = \begin{pmatrix} .2 & .5 & .3 \\ .2 & .5 & .3 \\ .2 & .5 & .2 \end{pmatrix}.$$

It is clear that $|P(t_2) - P(\infty)| < |P(t_1) - P(\infty)|$. We desire to develop scalar measures to reflect this situation.

Two measures that appear reasonable are the following:

$$\overline{M}_1(t) = M_1(t)/M_1(0) ,$$

where

$$M_1(t) = \sum_{i=0}^{M+y} \pi_i \sum_{j=0}^{M+y} |p_{ij}(t) - \pi_j| \tag{17}$$

and

$$\overline{M}_2(t) = M_2(t)/M_2(0) ,$$

where

$$M_2(t) = \sum_{i=0}^{M+y} \sum_{j=0}^{M+y} |p_{ij}(t) - \pi_j| . \tag{18}$$

Both measures are normalized to vary between 0 and 1 $[\overline{M}_1(0) = 1$, $M_1(\infty) = 0]$. The second is merely the sum of all the absolute differences between the corresponding elements of $P(t)$ and $P(\infty)$. The first is a weighted sum of the corresponding element absolute differences, the weights being the steady state probabilities; that is,

errors in the state probabilities which are seldom visited in steady state are "discounted."

A well-known mathematical definition of the distance between two matrices is the norm of the matrix formed by subtracting corresponding elements, denoted by $||P(t) - P(\infty)||$. Matrix norms must obey certain properties (see Reference [7], page 59), which are given below.

(1) $||P(t) - P(\infty)|| \geqq 0$ and $||P(t) - P(\infty)|| = 0$, iff $P(t) - P(\infty) = 0$;

(2) $||a[P(t) - P(\infty)]|| = |a| \cdot ||P(t) - P(\infty)||$;

(3) $||[P(t_1) - P(\infty)] + [P(t_2) - P(\infty)]|| \leqq ||P(t_1) - P(\infty)|| + ||P(t_2) - P(\infty)||$;

(4) $||[P(t_1) - P(\infty)] \cdot [P(t_2) - P(\infty)]|| \leqq ||P(t_1) - P(\infty)|| \cdot ||P(t_2) - P(\infty)||$.

We have not been able to show that $\overline{M}_1$ and $\overline{M}_2$ obey all the above properties, although we believe that they do. However, variations of $\overline{M}_1$ and $\overline{M}_2$ which do obey all properties are (see Reference [7], page 61):

$$\overline{M}_3(t) = M_3(t)/M_3(0)$$

where

$$M_3(t) = \sum_{i=0}^{M+y} \pi_i \max_j |p_{ij}(t) - \pi_j| \tag{19}$$

and

$$\overline{M}_4(t) = M_4(t)/M_4(0)$$

where

$$M_4(t) = \sum_{j=0}^{M+y} \max_i |p_{ij}(t) - \pi_j| \; . \tag{20}$$

Once again, $\overline{M}_3$ uses the π_i to weight the maximum row deviations, while $\overline{M}_4$ is based on the unweighted maximum column deviations. We have run many cases to observe the behavior of the proposed measures.

We chose situations for which $M+y$ equaled five so that we could view the entire 6×6 $P(t)$ matrix and compare it with the 6×6 $P(\infty)$ matrix. The cases run are shown in Table II, with the results shown in Table III.

TABLE II

6×6 CASES RUN

Case	M	y	c	λ	μ	"ρ" = Mλ/cμ
1	3	2	2	.2	0.5	0.6
2	3	2	2	.4	1.0	0.6
3	3	2	2	.6	1.5	0.6
4	3	2	2	.8	2.0	0.6
5	3	2	2	.2	1.0	0.3
6	3	2	2	.6	1.0	0.9
7	3	2	2	.8	1.0	1.2
8	3	2	2	.2	0.33	0.9
9	3	2	2	.2	0.25	1.2
10	3	2	1	.2	1.0	0.6
11	4	1	1	.15	1.0	0.6
12	4	1	2	.15	0.5	0.6

Before comparing the four measures numerically, we would like to present two properties that may help in judging the measures.

Property 1: $$\sum_{j=0}^{M+Y} |p_{ij}(t) - \pi_j| = 2 \sum_{j=j_\ell(i,t)}^{j_r(i,t)} (p_{ij}(t) - \pi_j) .$$

This follows from Theorem 2, which yields

$$\sum_{j=0}^{M+Y} |p_{ij}(t) - \pi_j| = \sum_{j=j_\ell(i,t)}^{j_r(i,t)} (p_{ij}(t) - \pi_j) + \sum_{j<j_\ell(i,t)} (\pi_j - p_{ij}(t))$$

$$+ \sum_{j>j_r(i,t)} (\pi_j - p_{ij}(t)) ;$$

TABLE III

RESULTS OF 6×6 CASES RUN

Case	A(∞)	Measures	t=0	t=1	t=3	t=5	t=10
1	.820	$\overline{M}_1$	1.000	.376	.143	.066	.010
		$\overline{M}_2$	1.000	.505	.232	.115	.019
		$\overline{M}_3$	1.000	.319	.103	.046	.007
		$\overline{M}_4$	1.000	.366	.178	.092	.016
2	.820	$\overline{M}_1$	1.000	.216	.046	.010	.001
		$\overline{M}_2$	1.000	.335	.081	.019	.001
		$\overline{M}_3$	1.000	.170	.031	.007	.000
		$\overline{M}_4$	1.000	.246	.066	.016	.000
3	.820	$\overline{M}_1$	1.000	.142	.015	.002	.001
		$\overline{M}_2$	1.000	.232	.027	.003	.000
		$\overline{M}_3$	1.000	.103	.010	.001	.000
		$\overline{M}_4$	1.000	.178	.023	.003	.000
4	.820	$\overline{M}_1$	1.000	.097	.005	.001	.001
		$\overline{M}_2$	1.000	.164	.009	.001	.001
		$\overline{M}_3$	1.000	.068	.003	.000	.000
		$\overline{M}_4$	1.000	.129	.008	.000	.000
5	.964	$\overline{M}_1$	1.000	.266	.039	.007	.001
		$\overline{M}_2$	1.000	.464	.110	.025	.001
		$\overline{M}_3$	1.000	.262	.039	.007	.001
		$\overline{M}_4$	1.000	.325	.092	.022	.000
6	.630	$\overline{M}_1$	1.000	.221	.043	.009	.001
		$\overline{M}_2$	1.000	.275	.056	.012	.001
		$\overline{M}_3$	1.000	.120	.020	.004	.000
		$\overline{M}_4$	1.000	.099	.040	.009	.000

TABLE III--*continued*

Case	A(∞)	Measures	t=0	t=1	t=3	t=5	t=10
7	.462	$\bar{M}_1$	1.000	.198	.033	.006	.000
		$\bar{M}_2$	1.000	.237	.041	.007	.000
		$\bar{M}_3$	1.000	.103	.016	.003	.000
		$\bar{M}_4$	1.000	.164	.023	.004	.000
8	.624	$\bar{M}_1$	1.000	.445	.223	.123	.034
		$\bar{M}_2$	1.000	.515	.276	.159	.044
		$\bar{M}_3$	1.000	.371	.120	.062	.016
		$\bar{M}_4$	1.000	.417	.199	.111	.031
9	.462	$\bar{M}_1$	1.000	.478	.258	.154	.052
		$\bar{M}_2$	1.000	.536	.304	.186	.063
		$\bar{M}_3$	1.000	.416	.145	.077	.025
		$\bar{M}_4$	1.000	.457	.215	.125	.037
10	.860	$\bar{M}_1$	1.000	.371	.155	.076	.013
		$\bar{M}_2$	1.000	.522	.268	.141	.026
		$\bar{M}_3$	1.000	.332	.128	.061	.011
		$\bar{M}_4$	1.000	.384	.205	.114	.022
11	.735	$\bar{M}_1$	1.000	.349	.121	.053	.007
		$\bar{M}_2$	1.000	.535	.260	.127	.019
		$\bar{M}_3$	1.000	.319	.105	.044	.006
		$\bar{M}_4$	1.000	.400	.211	.111	.017
12	.656	$\bar{M}_1$	1.000	.359	.114	.048	.006
		$\bar{M}_2$	1.000	.521	.227	.106	.014
		$\bar{M}_3$	1.000	.315	.092	.036	.004
		$\bar{M}_4$	1.000	.376	.186	.092	.013

and as

$$1 = \sum_j p_{ij}(t) = \sum_j \pi_j ,$$

then

$$\sum_{j=0}^{M+Y} |p_{ij}(t) - \pi_j| = 2 \sum_{j=j_\ell(i,t)}^{j_r(i,t)} (p_{ij}(t) - \pi_j) .$$

Property 1 suggests that $\overline{M}_1(t)$ and $\overline{M}_2(t)$ may in general require less computation, since only part of the matrix $P(t)$ has to be calculated. Further, while all numerical work indicated that $j_\ell(i,t)$ and $j_r(i,t)$ do not decrease as i increases (that is, the "clump," $p_{ij}(t) - \pi_j \geqq 0$, moves to the right as we go down rows of the $P(t)$ matrix), we have not been able to prove this.

Property 2: The measures $M_1(t)$, $M_2(t)$, and $M_4(t)$ decrease monotonically in t.

To show that $\overline{M}_1(t)$ and $\overline{M}_2(t)$ are monotone, we use Expression (16) to obtain for $j_\ell(i,t) \neq 0$ and $j_r(i,t) \neq M+Y$ that

$$\begin{aligned} \frac{d}{dt} \sum_{j=j_\ell(i,t)}^{j_r(i,t)} (p_{ij}(t) - \pi_j) &= \lambda_{j_\ell(i,t)-1}\left(p_{i,j_\ell(i,t)-1}(t) - \pi_{j_\ell(i,t)-1}\right) \\ &- \mu_{j_\ell(i,t)}\left(p_{i,j_\ell(i,t)}(t) - \pi_{j_\ell(i,t)}\right) \\ &- \lambda_{j_r(i,t)}\left(p_{i,j_r(i,t)}(t) - \pi_{j_r(i,t)}\right) \\ &+ \mu_{j_r(i,t)+1}\left(p_{i,j_r(i,t)+1}(t) - \pi_{j_r(i,t)+1}\right) . \end{aligned} \tag{21}$$

Expression (21) yields the monotone convergence since all four terms of the RHS are nonpositive. The results for the cases $j_\ell(i,t) = 0$ and $j_r(i,t) = M+Y$ can be obtained in a similar way.

We now show that $\overline{M}_4(t)$ is monotone. First we notice that

$$\max_i |p_{ij}(t) - \pi_j| = \max\left\{\max_i\{p_{ij}(t)\} - \pi_j \; ; \; -\left(\min_i\{p_{ij}(t)\} - \pi_j\right)\right\} .$$

Thus the statement is true if $\max_i\{p_{ij}(t)\}$ is decreasing and $\min_i\{p_{ij}(t)\}$ is increasing. These requirements are met since

$$P(t_1+t_2) = P(t_1)P(t_2) \; ;$$

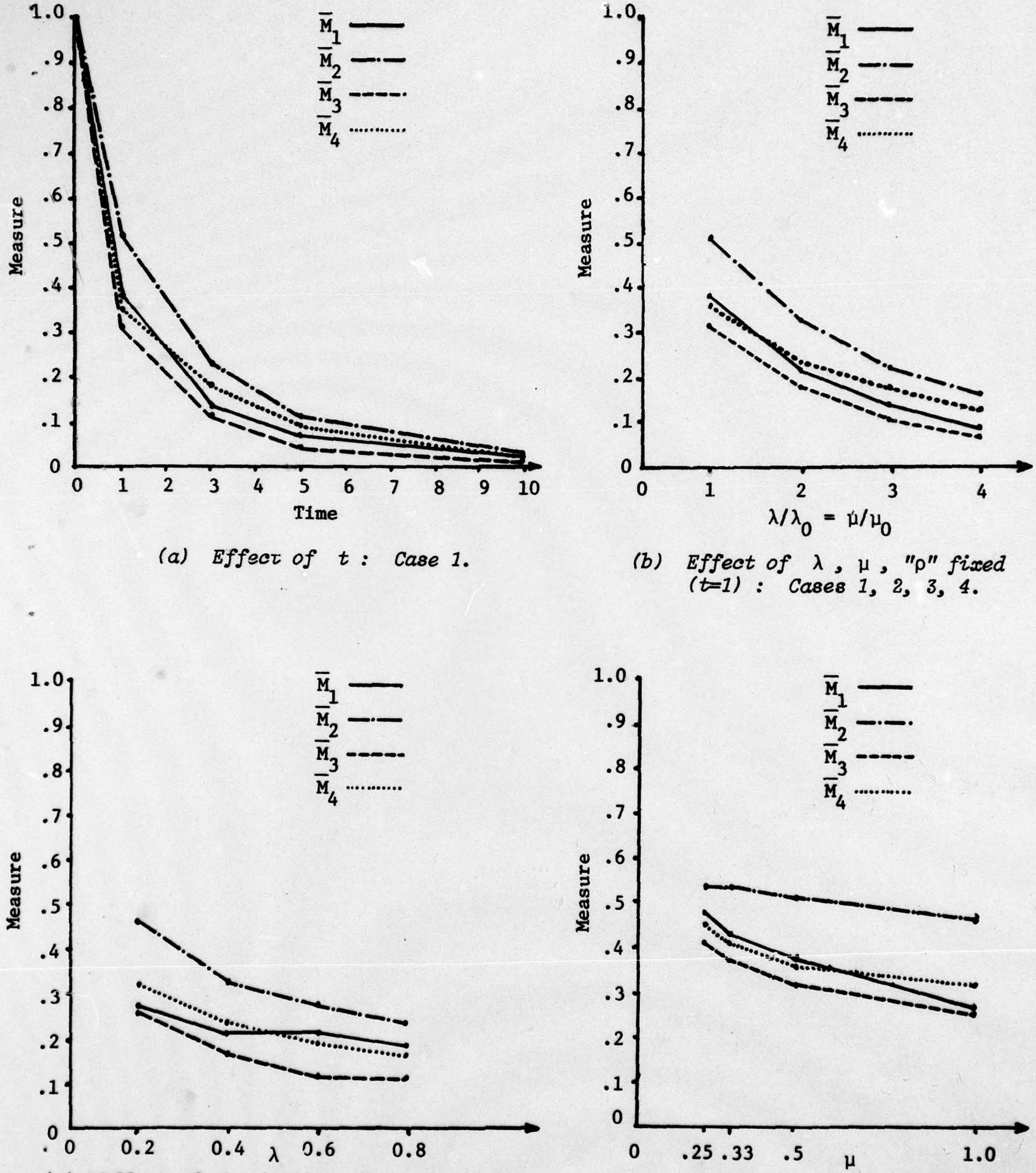
hence the elements of the jth column of $P(t_1+t_2)$ are convex combinations of the jth column of $P(t_2)$.

As for $\overline{M}_3(t)$, in all our calculations this measure was monotone decreasing. However, we are unable to prove it.

Since we are interested in assessing the relative merits of all the measures, for the cases in Table II all $p_{ij}(t)$ were calculated.

Case 1 can be considered the base case. Cases 2, 3, 4 increase λ and μ , keeping ρ (defined as $M\lambda/c\mu$) constant. Cases 5, 6, and 7 have μ constant but varying λ , while cases 8, 9, and 10 have λ constant with varying μ . Cases 11 and 12 revert back to the base case ρ , but vary M , y , and c .

Figure 4 shows some graphs of the measures as they vary with certain parameters. In 4(a), we see that all measures do decrease monotonically with time (in fact, they appear convex), which certainly must be a requirement for the measures to be useful. The unweighted measure $\overline{M}_2$ decreases more slowly than its weighted counterpart, $\overline{M}_1$. The same is true of $\overline{M}_4$ and $\overline{M}_3$. Of course, $\overline{M}_3$ is always less than $\overline{M}_1$, and $\overline{M}_4$ is always less than $\overline{M}_2$, since $\overline{M}_3$ and $\overline{M}_4$ consider only maximum row and column errors, respectively.



(a) Effect of t : Case 1.

(b) Effect of λ , μ , "ρ" fixed (t=1) : Cases 1, 2, 3, 4.

(c) Effect of λ (t=1) : Cases 2, 5, 6, 7.

(d) Effect of μ (t=1) : Cases 1, 5, 8, 9.

Figure 4.--Results of 6×6 runs.

In Figure 4(b), we see that increasing λ and μ together so as to keep ρ constant causes a faster approach to steady state (all measures again decrease monotonically). This is intuitive, since we would expect the greater the transition (arrival and service completion) rate, the faster steady state should be approached. While this generally holds also for varying only arrival rate (service completion rate) keeping service completion rate (arrival rate) fixed [see Figures 4(c) and 4(d)], the measures do not always decrease monotonically. In the case where μ is fixed, $\overline{M}_1$ decreases, then slightly increases, then decreases again as λ increases. In other runs not shown in Table III we saw similar effects for the other measures. Just why this is so is not intuitively clear, although one must keep in mind that the traffic intensity (ρ) is now also varying, so that the probability distributions of the system states are also changing.

The final two cases run (Cases 11 and 12) had the same ρ but varied M, y, and c. Even with ρ fixed, $A(\infty)$ varies, showing the effect of population size, spares pool size, and repair capacity on availability. However, the convergence to steady state as given by the four measures does not appear to change significantly.

Which measure to choose and how to utilize the measure ultimately lie in a subjective judgment. We see that $\overline{M}_2$ decreases more slowly than the others, while $\overline{M}_3$ decreases most rapidly. Since in effect the scale is arbitrary, one a measure (or measures) is chosen, subjective benchmarks would have to be set up. For example, consider the $P(t)$ matrices for Case 1, $t=1,3,5$ and ∞, and the respective measures shown in Figure 5. The full $P(t)$ matrices for $t=1,3,5$ are given. At the bottom of each is listed the steady state probability, π_j [the $P(\infty)$ matrix has all its rows identical and equal to $\pi = (\pi_0, \pi_1, \ldots, \pi_{M+Y})$ so that near steady state all elements in column j of a $P(t)$ matrix should be close to π_j].

	P(1)	P(3)	P(5)
	$\begin{bmatrix} .62 & .29 & .07 & .01 & .00 & .00 \\ .24 & .49 & .20 & .05 & .01 & .00 \\ .10 & .33 & .36 & .18 & .03 & .00 \\ .03 & .15 & .29 & .38 & .13 & .01 \\ .01 & .05 & .14 & .33 & .40 & .07 \\ .00 & .01 & .05 & .17 & .36 & .40 \end{bmatrix}$	$\begin{bmatrix} .39 & .36 & .16 & .07 & .02 & .00 \\ .30 & .36 & .20 & .10 & .03 & .00 \\ .23 & .33 & .23 & .15 & .06 & .01 \\ .15 & .28 & .24 & .20 & .10 & .02 \\ .09 & .21 & .23 & .25 & .16 & .05 \\ .05 & .15 & .20 & .27 & .24 & .10 \end{bmatrix}$	$\begin{bmatrix} .33 & .35 & .19 & .10 & .03 & .01 \\ .29 & .34 & .20 & .11 & .04 & .01 \\ .26 & .33 & .21 & .13 & .05 & .01 \\ .22 & .31 & .22 & .16 & .07 & .02 \\ .18 & .28 & .22 & .18 & .10 & .03 \\ .14 & .25 & .22 & .21 & .14 & .04 \end{bmatrix}$
$P(\infty)$:	.28 .34 .20 .12 .05 .01	.28 .34 .20 .12 .05 .01	.28 .34 .20 .12 .05 .01
$\overline{M}_1$:	.376	.143	.066
$\overline{M}_2$:	.505	.232	.115
$\overline{M}_3$:	.319	.103	.046
$\overline{M}_4$:	.366	.178	.092
$A(t)_{UP}$:	.917	.917	.871
$A(t)_{DN}$:	.064	.393	.612
$A(\infty)$:	.820	.820	.820

Figure 5.--$P(t)$, $\overline{M}_i(t)$ and $A(t)$ for Case 1.

In viewing Figure 5, it is fairly clear that $P(1)$ is extremely far from $P(\infty)$ and $P(3)$ is quite far, while $P(5)$ is somewhat closer--although in a few places there are some sizable errors, particularly in rows 5 and 6, which, of course, are discounted by measures $\bar{M}_1$ and $\bar{M}_3$. Thus one might conclude that a value of $\bar{M}_1$ of .05 or less indicates a near steady state condition, while a value for $\bar{M}_2$ of .10 or less suffices.

Also shown in Figure 5 is the steady state population availability, $A(\infty)$, along with two transient $A(t)$'s. One of these, $A(t)_{UP}$, assumes all units are operational at time zero, and the other, $A(t)_{DN}$, assumes the opposite, namely, all units are down (in or awaiting repair). Note, as before, that $A(t)_{UP}$ is much closer to $A(\infty)$ than is $A(t)_{DN}$, since $A(\infty)$ is nearer to $A(0)_{UP} = 1$ than to $A(0)_{DN} = 0$.

As to which measure might be preferable, again subjectivity is required. Consider Figure 6, where $P(1)$ for Cases 1, 11, and 12 is shown, along with the respective $P(\infty)$, $A(t)$, and $A(\infty)$. In viewing the matrices, it appears that all are about the same distance from steady state. In looking at $A(t)_{UP}$ and $A(t)_{DN}$ versus $A(\infty)$, perhaps one might say that Case 12 is the "poorest" case, while Case 1 is the "best." None of the measures indicates Case 12 to be the poorest, although $\bar{M}_2$ and $\bar{M}_4$ come closest to this. The measures $\bar{M}_2$ and $\bar{M}_4$ do indicate that Case 1 is the best.

In Figure 7 two more cases are compared, namely, Cases 2 and 6. Again it is difficult to judge which $P(1)$ is the better, although in looking at the $A(1)$'s one might give "the edge" to Case 2. Here, however, $\bar{M}_1$ shows Case 2 to be preferable, while $\bar{M}_2$, $\bar{M}_3$, and $\bar{M}_4$ show the opposite.

It appears that no measure is superior, so that the choice should perhaps be the one that is computationally easiest to obtain. In view of

	Case 1 P(1)						Case 11 P(1)						Case 12 P(1)					
	.62	.29	.07	.01	.00	.00	.68	.25	.06	.01	.00	.00	.62	.29	.07	.01	.00	.00
	.24	.49	.20	.05	.01	.00	.41	.37	.17	.03	.00	.00	.24	.50	.21	.04	.00	.00
	.10	.33	.36	.18	.03	.00	.17	.29	.37	.14	.02	.00	.10	.35	.39	.14	.02	.00
	.03	.15	.29	.38	.13	.01	.05	.14	.31	.38	.10	.00	.03	.16	.32	.38	.10	.01
	.01	.05	.14	.33	.40	.07	.01	.04	.15	.34	.39	.06	.01	.05	.15	.34	.39	.06
	.00	.01	.05	.17	.36	.40	.00	.01	.05	.17	.37	.40	.00	.01	.05	.17	.37	.40
$P(\infty)$:	.28	.34	.20	.12	.05	.01	.46	.28	.17	.07	.02	.00	.30	.36	.21	.10	.03	.00
$\overline{M}_1$:			.376						.349						.359			
$\overline{M}_2$:			.505						.535						.521			
$\overline{M}_3$:			.319						.319						.315			
$\overline{M}_4$:			.366						.400						.376			
$A(t)_{UP}$:			.917						.928						.920			
$A(t)_{DN}$:			.064						.013						.014			
$A(\infty)$:			.820						.714						.656			

Figure 6.--$P(1)$, $\overline{M}_i(1)$ and $A(1)$ for Cases 1, 11, 12.

	Case 2	Case 6
P(1)	$\begin{bmatrix} .47 & .35 & .13 & .04 & .01 & .00 \\ .29 & .39 & .20 & .09 & 02 & .00 \\ .18 & .34 & .26 & .16 & .05 & .01 \\ .09 & .24 & .27 & .25 & .12 & .02 \\ .04 & .14 & .22 & .30 & .23 & .06 \\ .02 & .07 & .14 & .26 & .32 & .19 \end{bmatrix}$	$\begin{bmatrix} .32 & .36 & .20 & .09 & .03 & .00 \\ .20 & .34 & .25 & .15 & .05 & .01 \\ .12 & .27 & .26 & .22 & .10 & .02 \\ .06 & .19 & .24 & .28 & .18 & .05 \\ .03 & .11 & .19 & .30 & .27 & .10 \\ .01 & .05 & .12 & .25 & .33 & .22 \end{bmatrix}$
$P(\infty)$:	.28 .34 .20 .12 .05 .01	.14 .26 .23 .21 .12 .04
$\overline{M}_1$:	.216	.221
$\overline{M}_2$:	.335	.275
$\overline{M}_3$:	.170	.120
$\overline{M}_4$:	.246	.199
$A(1)_{UP}$:	.951	.879
$A(1)_{DN}$:	.229	.193
$A(\infty)$:	.820	.630

Figure 7.--P(1) , $\overline{M}_i(1)$ and A(1) for Cases 2 and 6.

the theorems, $\overline{M}_1$ and $\overline{M}_2$ are computationally most efficient for large matrices, so that either of these might be an appropriate choice.

REFERENCES

[1] BARZILY, Z., D. GROSS, and H. D. KAHN (1977). Some practical considerations in the application of finite source queueing models. Technical Paper Serial T-360. Program in Logistics, The George Washington University.

[2] CINLAR, E. (1975). *Introduction to Stochastic Processes*, Prentice-Hall, New Jersey.

[3] COHEN, J. W. (1969). *The Single Server Queue*. North-Holland, Amsterdam; Wiley, New York.

[4] GRASSMAN, W. (1977). Transient solutions in Markovian queueing systems. *Computers and Operations Res.* **4** 47-56.

[5] GROSS, D. and C. M. HARRIS (1974). *Fundamentals of Queueing Theory*. Wiley, New York.

[6] GROSS, D., H. D. KAHN, and J. D. MARSH (1977). Queueing models for spares provisioning. *Naval Res. Logist. Quart.* **24** 521-536.

[7] MARLOW, W. H. (1978). *Mathematics for Operations Research*. Wiley, New York.

THE GEORGE WASHINGTON UNIVERSITY
Program in Logistics
Distribution List for Technical Papers

The George Washington University
  Office of Sponsored Research
  Library
  Vice President H. F. Bright
  Dean Harold Liebowitz
  Dean Henry Solomon

ONR
  Chief of Naval Research
   (Codes 200, 434)
  Resident Representative

OPNAV
  OP-40
  DCNO, Logistics
  Navy Dept Library
  NAVDATA Automation Cmd
  OP-964

Naval Aviation Integrated Log Support

NARDAC Tech Library

Naval Electronics Lab Library

Naval Facilities Eng Cmd Tech Library

Naval Ordnance Station
  Louisville, Ky.
  Indian Head, Md.

Naval Ordnance Sys Cmd Library

Naval Research Branch Office
  Boston
  Chicago
  New York
  Pasadena
  San Francisco

Naval Ship Eng Center
  Philadelphia, Pa.
  Washington, DC

Naval Ship Res & Dev Center

Naval Sea Systems Command
  PMS 30611
  Tech Library
  Code 073

Naval Supply Systems Command
  Library
  Operations and Inventory Analysis

Naval War College Library
  Newport

BUPERS Tech Library

FMSO

Integrated Sea Lift Study

USN Ammo Depot Earle

USN Postgrad School Monterey
  Library
  Dr Jack R. Borsting
  Prof C. R. Jones

US Marine Corps
  Commandant
  Deputy Chief of Staff, R&D

Marine Corps School Quantico
  Landing Force Dev Ctr
  Logistics Officer
  Commanding Officer
   USS Francis Marion (LPA-249)

Armed Forces Industrial College

Armed Forces Staff College

Army War College Library
  Carlisle Barracks

Army Cmd & Gen Staff College

Army Logistics Mgt Center
  Fort Lee

Commanding Officer, USALDSRA
  New Cumberland Army Depot

Army Inventory Res Ofc
  Philadelphia

Air Force Headquarters
  AFADS-3
  LEXY
  SAF/ALG

Griffiss Air Force Base
  Reliability Analysis Center

Gunter Air Force Base
  AFLMC/XR

Maxwell Air Force Base Library

Wright-Patterson Air Force Base
  Log Command
  Research Sch Log
  AFALD/XR

Defense Documentation Center

National Academy of Sciences
  Maritime Transportation Res Board Library

National Bureau of Standards
  Dr B. H. Colvin
  Dr Joan Rosenblatt

National Science Foundation

National Security Agency

Weapon Systems Evaluation Group

British Navy Staff

National Defense Hdqtrs, Ottawa
  Logistics, OR Analysis Establishment

American Power Jet Co
  George Chernowitz

General Dynamics, Pomona

General Research Corp
  Dr Hugh Cole
  Library

Logistics Management Institute
  Dr Murray A. Geisler

MATHTEC
  Dr Eliot Feldman

Rand Corporation
  Library

Carnegie-Mellon University
  Dean H. A. Simon
  Prof G. Thompson

Case Western Reserve University
  Prof B. V. Dean
  Prof M. Mesarovic
  Prof S. Zacks

Cornell University
  Prof R. E. Bechhofer
  Prof R. W. Conway
  Prof Andrew Schultz, Jr.

Cowles Foundation for Research in Economics
  Prof Herbert Scarf
  Prof Martin Shubik

Florida State University
  Prof R. A. Bradley

Harvard University
  Prof K. J. Arrow
  Prof W. G. Cochran
  Prof Arthur Schleifer, Jr.

Princeton University
  Prof A. W. Tucker
  Prof J. W. Tukey
  Prof Geoffrey S. Watson

Purdue University
Prof S. S. Gupta
Prof H. Rubin
Prof Andrew Whinston

Stanford University
Prof T. W. Anderson
Prof G. B. Dantzig
Prof F. S. Hillier
Prof D. L. Iglehart
Prof Samuel Karlin
Prof G. J. Lieberman
Prof Herbert Solomon
Prof A. F. Veinott, Jr.

University of California, Berkeley
Prof R. E. Barlow
Prof D. Gale
Prof Jack Kiefer
Prof Rosedith Sitgreaves

University of California, Los Angeles
Prof J. R. Jackson
Prof R. R. O'Neill

University of North Carolina
Prof W. L. Smith
Prof M. R. Leadbetter

University of Pennsylvania
Prof Russell Ackoff
Prof Thomas L. Saaty

University of Texas
Prof A. Charnes

Yale University
Prof F. J. Anscombe
Prof I. R. Savage

Prof Z. W. Birnbaum
University of Washington

Prof B. H. Bissinger
The Pennsylvania State University

Prof Seth Bonder
University of Michigan

Prof G. E. P. Box
University of Wisconsin

Dr Jerome Bracken
Institute for Defense Analyses

Prof H. Chernoff
Mass. Institute of Technology

Prof Arthur Cohen
Rutgers - The State University

Mr Wallace M. Cohen
US General Accounting Office

Prof C. Derman
Columbia University

Prof Masao Fukushima
Kyoto University

Prof Saul I. Gass
University of Maryland

Dr Donald P. Gaver
Carmel, California

Prof Amrit L. Goel
Syracuse University

Prof J. F. Hannan
Michigan State University

Prof H. O. Hartley
Texas A & M Foundation

Mr Gerald F. Hein
NASA, Lewis Research Center

Prof W. M. Hirsch
Courant Institute

Dr Alan J. Hoffman
IBM, Yorktown Heights

Prof John R. Isbell
State University of New York, Amherst

Dr J. L. Jain
University of Delhi

Prof J. H. K. Kao
Polytech Institute of New York

Prof W. Kruskal
University of Chicago

Mr S. Kumar
University of Madras

Prof C. E. Lemke
Rensselaer Polytech Institute

Prof Loynes
University of Sheffield, England

Prof Steven Nahmias
University of Pittsburgh

Prof D. B. Owen
Southern Methodist University

Prof E. Parzen
Texas A & M University

Prof H. O. Posten
University of Connecticut

Prof R. Remage, Jr.
University of Delaware

Prof Hans Riedwyl
University of Bern

Dr Fred Rigby
Texas Tech College

Mr David Rosenblatt
Washington, D. C.

Prof M. Rosenblatt
University of California, San Diego

Prof Alan J. Rowe
University of Southern California

Prof A. H. Rubenstein
Northwestern University

Dr M. E. Salveson
West Los Angeles

Prof Edward A. Silver
University of Waterloo, Canada

Prof M. J. Sobel
Georgia Inst of Technology

Prof R. M. Thrall
Rice University

Dr S. Vajda
University of Sussex, England

Prof T. M. Whitin
Wesleyan University

Prof Jacob Wolfowitz
University of South Florida

Prof Max A. Woodbury
Duke University

December 1978